

# Measure of entanglement by Singular Value decomposition

Nikolay Raychev

**Abstract** This report describes an approach for representation of quantum entanglement through state matrix. The proposed approach could be used to allow encoding of more information. The Singular Value Decomposition is useful as a measure for entanglement because the unitary operations preserve it.

**Keywords** Quantum operators, entanglement, Phase, Quantum circuit



## 1. INTRODUCTION

In the development process of a circuit quantum simulator [10, 11, 12] was required to seek solutions for representation of quantum entanglement. The study on the actual mathematics expressing the entanglement is much more enlightening than reading of popular scientific expositions for it. However, the question remains how to think about the entanglement in a useful way when solving problems. In this article is described an interesting approach for perception of the entanglement. The developed by the author of this article quantum circuit simulator enables dragging and dropping quantum gates on the circuit and displays presentations of the output state. Each quantum system is described with a bunch of complex weights, one for each basis state. To describe a system with  $n$  qubits are needed  $2^n$  weights, because it has  $2^n$  basis (i.e. classical) states. This makes the visual presentation of quantum states with more than a few qubits difficult, because there are so many numbers. For visual representation of the weights at the output of the circuit in the simulator can be used a vector-column. Each complex number is presented with an oriented circle, as they are arranged vertically in binary order (i.e. from  $|0000\rangle$  to  $|1111\rangle$ ). This leads to problems, even with four qubits. It is not only difficult to determine which weight is located where, but also the available space is used poorly due to the column being thin. Therefore, it is better to rearrange the column in a grid.

The arrangement in a grid turned out to be a better solution. Suddenly, the operations on half of the qubits have row-wise effects, while the operations on the other half - column-wise effects. This can be used as a mnemonic tool.

When working with circuits, which include two parts, the Side A's effects are distinguished from Side B's effects by submitting to Side A row-wise qubits, and to Side B - column-wise qubits. In the end it is noticed that the grid with the states acts as a matrix. If a complex entangled pair of qubits is created and a gate is applied to one of the qubits, the state would look like as a matrix for this gate. And the addition of another gate multiplies "the matrix" by the next gate. This helps to clarify how the broken symmetry will look.

## 2. STATE TIMES OPERATION

Let's examine the state  $S_0 = a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle$ , represented as the matrix  $Grid(S_0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . What will happen when applying the operation  $U = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$  to the first qubit of  $S_0$ ? (The first qubit is the row-wise one.)

After performing the computations:

$$\begin{aligned} S_1 &= (U \otimes I) \cdot S_0 \\ &= \left( U \begin{bmatrix} a \\ b \end{bmatrix} \right) \otimes |0\rangle + \left( U \begin{bmatrix} c \\ d \end{bmatrix} \right) \otimes |1\rangle \end{aligned}$$

$$\begin{aligned}
 &= \begin{bmatrix} ea + bf \\ ag + bh \end{bmatrix} \otimes |0\rangle + \begin{bmatrix} ec + df \\ cg + dh \end{bmatrix} \otimes |1\rangle \\
 &= (ea + bf)|00\rangle + (ag + bh)|10\rangle + (ec + df)|01\rangle + (cg + dh)|11\rangle
 \end{aligned}$$

By rearranging the final state into a matrix is established that  $Grid(S_1) = \begin{bmatrix} ea + bf & ag + bh \\ ec + df & cg + dh \end{bmatrix}$ . This is also equal to  $Grid(S_0) \cdot U^T$ . The results of applying a single qubit operation can be computed while avoiding the tensor product. The same phenomenon occurs when applying  $U$  to the second qubit, except that there is no transpose and the multiplication occurs on the other side:  $Grid(S_2) = U \cdot Grid(S_0)$

By representing the state as a matrix the application of operations is simplified. The operating on the first qubit (the row-wise one) is equivalent to subsequent multiplication of the state matrix by transpose of the operation. The operating on the second qubit (the column-wise one) is equivalent to preliminary multiplication of the state matrix by the operation. Also multi-qubit operations can be represented (for example controlled NOT-s, corresponding to flipping the bottom row or the right column), but they will not be used in this article.

Particularly useful and convenient is that the operations are simplified in such a way. The reasoning for the simplified operations makes it obvious why the operations on different qubits must be reduced (they are accumulated on opposite sides). This facilitates also the understanding about the states that can be reached from any starting state ... Let's consider how the single qubits operations behave when starting from two different states: an unentangled classical "Single corner" state  $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and a completely entangled "shared diagonal" state  $\begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & \sqrt{0.5} \end{bmatrix}$ .

### 3. SINGLE CORNER

When starting in the matrix state  $S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  what types of states can be reached by operating independently on the first and second qubits?

With the writing of a test code can be experimented and to be seen what will be achieved by rotating and phasing both qubits. In particular, can be checked whether it will reach the entangled shared diagonal state:

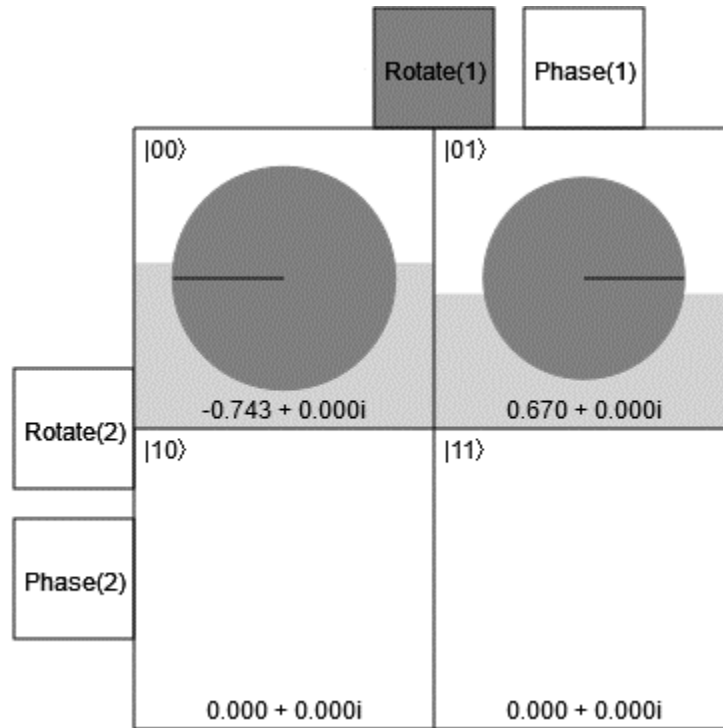


Figure 1 Single Corner

If  $U_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is applied to the first qubit and  $U_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$  to the second qubit (always multiple operations can be merged by multiplying them together), is obtained:

$$\begin{aligned}
 S_f &= U_2 S_0 U_1^T \\
 &= \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
 &= \begin{bmatrix} e & 0 \\ g & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
 &= \begin{bmatrix} ae & be \\ ag & bg \end{bmatrix} \\
 &= \begin{bmatrix} e \\ g \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}
 \end{aligned}$$

It is found that 1) two of the four coefficients from each operation are eliminated, and that 2) the resulting state matrix can be described with just two complex ratios (plus a global phase factor that is irrelevant). The horizontal ratio is determined by operations on the first qubit, and the vertical ratio - by operations on the second qubit

Examples of the types of states, which can be reached, include

$$\begin{bmatrix} \sqrt{0.5} & \sqrt{0.5} \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0.6 & 0.8 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{0.5} & 0 \\ -\sqrt{0.5} & 0 \end{bmatrix}$$

$$S_0 = \begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & \sqrt{0.5} \end{bmatrix}.$$

It's impossible to reach the shared diagonal state from the Single corner state via independent operations, because the first state can't be represented by vertical and horizontal ratios

#### 4. SHARED DIAGONAL

Let's now consider the matrix state  $\begin{bmatrix} 0.5 & 0.5i \\ 0.5i & -0.5 \end{bmatrix}$ .

If checked more closely, will be noted, that the system behaves differently in a very few ways. For example, there are still states in which rotating the two qubits doesn't change the size of the circles much... but they're in a different place than they were in the Single corner case:

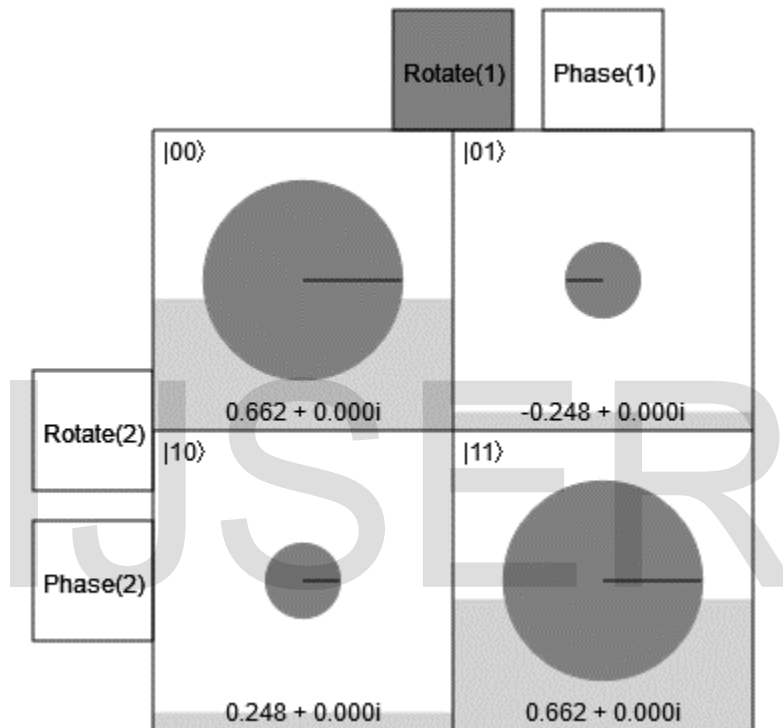


Figure 2 Shared diagonal

Let's analyze the situation algebraically, again. After applying  $U_1$  to the first qubit and  $U_2$  to the second qubit, is obtained:

$$\begin{aligned}
 S_f &= U_2 S_0 U_1^T \\
 &= U_2 \begin{bmatrix} \sqrt{0.5} & 0 \\ 0 & \sqrt{0.5} \end{bmatrix} U_1^T \\
 &= U_2 \sqrt{0.5} I U_1^T \\
 &= \sqrt{0.5} U_2 U_1^T
 \end{aligned}$$

The starting state is a unitary matrix (times  $\sqrt{0.5}$ ) and all the operations correspond to multiplying the state by a unitary matrix. So the final state will also be a unitary matrix (times  $\sqrt{0.5}$ ).

Unlike in the Single corner case, none of the operations' matrix coefficients are being removed. Also, the effects of the operations are no longer orthogonal. Instead of one qubit controlling the horizontal, and another - the vertical, they both control everything. Everything that the side *A* can make on the first qubit, the side *B* can undo by applying an appropriate counter-operation to the second qubit. Or the side *B* can apply the same effect as the side *A*, effectively squaring the operation. Or the side *A*, if she knows what the side *B* will do, can put the system into any (unitary) final state she desires.

Given the above facts, what is this interesting thing that could be done in the shared diagonal case, which can't be done in the Single corner case? The first is, that now Side *A* can put four numbers in the state instead of two. This would allow encoding of more information.

## CONCLUSION

In 2 qubit non-entangled systems, the state acts like a matrix composed by combining two complex ratios. The ratios are controlled independently, and orthogonally by operations on either side. In 2 qubit entangled systems, the state acts like a unitary matrix. All the matrix coefficients are controlled by both sides, with overlapping effects. The effects can be moved between the sides, if this is convenient when designing circuits. The sides can undo or square each others' effects. Systems that are partially entangled, e.g.  $\begin{bmatrix} 0.8 & 0 \\ 0 & 0.6 \end{bmatrix}$ , can be interpreted as a linear combination of non-entangled and entangled. The non-entangled and entangled states are like Singular Value Decomposition basis states. Larger systems, with multiple qubits per side (but still two sides), have more singular-value-decomposition basis states. The Singular Value Decomposition is useful as a measure for entanglement because the unitary operations preserve it. All of this breaks down, if the sides are more than two.

## REFERENCES

- [1] D. Deutsch, Proc. R. Soc. London A, 425, 73 (1989).
- [2] D. P. DiVincenzo, Phys. Rev. A 51,1015 (1995).
- [3] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, United Kingdom, 2000).
- [4] D. Loss and D. P. DiVincenzo, Phys. Rev. A 57, 120 (1998).
- [5] J. R. Petta, A. C. Johnson, J. M. Taylor, E. A. Laird, A. Yacoby, M. D. Lukin, C. M. Marcus, M. P. Hanson, and A. C. Gossard, Science 309, 2180 (2005).
- [6] D. P. DiVincenzo, D. Bacon, J. Kempe, G. Burkard, and K. B. Whaley, Nature (London) 408, 339 (2000).
- [7] B. H. Fong and S. M. Wandzura, Quantum Inf. Comput. 11, 1003 (2011).
- [8] P. Zanardi and M. Rasetti, Mod. Phys. Lett. B 11, 1085 (1997); Phys. Rev. Lett. 79, 3306 (1997).
- [9] L.-M. Duan and G.-C. Guo, Phys. Rev. A 57, 737 (1998).
- [10] Nikolay Raychev. (2015). Interactive environment for implementation and simulation of quantum algorithms. CompSysTech'15, DOI: 10.13140/RG.2.1.2984.3362
- [11] Nikolay Raychev. Unitary combinations of formalized classes in qubit space. International Journal of Scientific and Engineering Research 04/2015; 6(4):395-398. DOI: 10.14299/ijser.2015.04.003, 2015.
- [12] Nikolay Raychev. Functional composition of quantum functions. International Journal of Scientific and Engineering Research 04/2015; 6(4):413-415. DOI:10.14299/ijser.2015.04.004, 2015.
- [13] Nikolay Raychev. Logical sets of quantum operators. International Journal of Scientific and Engineering Research 04/2015; 6(4):391-394. DOI:10.14299/ijser.2015.04.002, 2015.
- [14] Nikolay Raychev. Controlled formalized operators. In International Journal of Scientific and Engineering Research 05/2015; 6(5):1467-1469, 2015.
- [15] Nikolay Raychev. Controlled formalized operators with multiple control bits. In International Journal of Scientific and Engineering Research 05/2015; 6(5):1470-1473, 2015.
- [16] Nikolay Raychev. Connecting sets of formalized operators. In International Journal of Scientific and Engineering Research 05/2015; 6(5):1474-1476, 2015.

- [17] Nikolay Raychev. Indexed formalized operators for n-bit circuits. International Journal of Scientific and Engineering Research 05/2015; 6(5):1477-1480, 2015.
- [18] Nikolay Raychev. Converting the transitions between quantum gates into rotations. International Journal of Scientific and Engineering Research 06/2015; 6(6): 1352-1354. DOI:10.14299/ijser.2015.06.001, 2015.
- [19] Nikolay Raychev. Quantum algorithm for non-local coordination. International Journal of Scientific and Engineering Research 06/2015; 6(6):1360-1364. DOI:10.14299/ijser.2015.06.003, 2015.
- [20] Nikolay Raychev. Universal quantum operators. International Journal of Scientific and Engineering Research 06/2015; 6(6):1369-1371. DOI:10.14299/ijser.2015.06.005, 2015.
- [21] Nikolay Raychev. Ensuring a spare quantum traffic. International Journal of Scientific and Engineering Research 06/2015; 6(6):1355-1359. DOI:10.14299/ijser.2015.06.002, 2015.
- [22] Nikolay Raychev. Quantum circuit for spatial optimization. International Journal of Scientific and Engineering Research 06/2015; 6(6):1365-1368. DOI:10.14299/ijser.2015.06.004, 2015.
- [23] Nikolay Raychev. Conflicts resolving by symmetry breaking. International Journal of Scientific and Engineering Research 05/2015; 6(5): 1481-1482. DOI:10.14299/ijser.2015.05.005, 2015.

IJSER